Lecture 13 (2/2/22).

Normal families in $\mathcal{M}(G)$.

Recall AA Thm.: $\mathcal{F}$ is normal in $C(G,\Omega)$ iff

(i) $\forall z \in G$, $\overline{\{f(z) : f \in \mathcal{F}\}} \subset\subset \Omega$.

(ii) $\forall z \in G$, $\mathcal{F}$ is equicont. at z.

Now, $\Omega = \mathbb{C}_\infty$, which is compact $\Rightarrow$ (i) is automatic. Thus, for $\Omega = \mathbb{C}_\infty$:

normal $\Leftrightarrow$ equicont. at every z

$\Leftrightarrow$ equicont. on every compact $K \subset\subset G$.
(Prop 1.22)

For motivation, let's examine the pf of Montel. To establish (ii) for a locally bdd family $\mathcal{F}$ in $H(G)$ we showed $\mathcal{F}$ locally bdd $\overset{\text{Cauchy's Estimate}}{\Longrightarrow}$ $\mathcal{F}' = \{f' : f \in \mathcal{F}\}$ was locally bdd.

Then, we used if $z, z' \in K = \overline{B(a,r)} \subseteq G$

$$|f(z') - f(z)| \leq \sup_K |f'| \, |z' - z| \quad (*)$$

to get equicont. on K.

The same argument, suitably interpreted, works for $\mathcal{F}$ in $\mathcal{M}(G)$. To obtain $(*)$, when measuring dist. between $f(z)$, $f(z')$ in $\mathbb{C}$, we used

$$|f(z) - f(z')| \leq \int_{f \circ \gamma_{z,z'}} |f'(\zeta)| \cdot |d\zeta| ,$$

where $\gamma_{z,z'} = [z, z']$. This is the Riemannian geometry approach: $d(w,w') = \inf_{\substack{\sigma \\ \sigma(0)=w \\ \sigma(1)=w'}} \int_0^1 |\sigma'(t)| dt$

$$= \inf_\sigma \int_\sigma ds ,$$

where $ds|_w = |dw|$, the infinitesimal Euclidian metric.

For distances in $\mathbb{C}_\infty$, we use instead the infinitesimal Fubini-Study metric:

$$ds\big|_w = \frac{2}{1+|w|^2}|dw| \quad \text{and}$$

$$d_\infty(w,w') = \inf_{\substack{\sigma \\ \text{curves } w \to w'}} \int_\sigma ds = \inf_{\substack{\sigma \\ \sigma(0)=w \\ \sigma(1)=w'}} \int_0^1 \frac{2|dw|}{1+|w|^2}$$

So, if you take $\sigma = f \circ \gamma_{z,z'}$, $\gamma_{z,z'} = [z,z']$ you get

$$d_\infty(f(z), f(z')) \leq \int_{[z,z']} \frac{2|f'(z)|}{1+|f(z)|^2}|dz| .$$

Thus, if we set $\mu(f)(z) = \dfrac{2|f'(z)|}{1+|f(z)|^2}$ and if $z, z' \in K = \overline{B(a,r)} \subset\subset G$,

$$d_\infty(f(z), f(z')) \leq \sup_K \mu(f) \, |z-z'| \quad (**).$$

So the condition $\mathcal{F}'$ locally bdd that we used in Montel to get equicont. should be replaced by $\mu(\mathcal{F}) = \{\mu(f) : f \in \mathcal{F}\}$ is locally bdd. This turns out to be the case also.

We shall give pf, as in Conway, without appealing to Riemannian geometry.

Def. ① For a meromorphic function f, let

$$\begin{cases} \mu(f)(a) = \dfrac{2|f'(a)|}{1+|f(a)|^2}, & \text{if } a \text{ not a pole} \\[2ex] \mu(f)(a) = \lim\limits_{z \to a} \mu(f)(z) = \mu\left(\dfrac{1}{f}\right)(a), & \text{if } a \text{ is a pole.} \end{cases}$$

Rem: • If a is a pole, $f(z) = \frac{A}{z^p} + O\left(\frac{1}{z^{p-1}}\right)$ ← Laurent series

then $\mu(f)(a) = \begin{cases} 0, & p \geq 2 \\ \frac{2}{|A|}, & p = 1. \end{cases}$

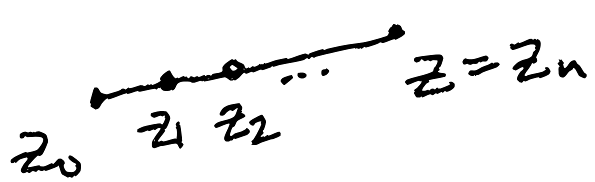
• Thus, if $f \in \mathcal{M}(G)$, then $\mu(f) \in C(G,\mathbb{C})$.

If $\mathcal{F}$ is family in $\mathcal{M}(G)$, $\mu(\mathcal{F})$ is the family in $C(G,\mathbb{C})$ s.t.

$$\mu(\mathcal{F}) = \{\mu(f) : f \in \mathcal{F}\}.$$

<u>Thm 1.</u> Let $\mathcal{F}$ be family in $\mathcal{M}(G)$. Then: $\mathcal{F}$ normal $\Leftrightarrow$ $\mu(\mathcal{F})$ locally bounded.

<u>Pf.</u> "$\Leftarrow$". As mentioned above, to apply AA we need only to verify that $\mathcal{F}$ is equicont. at every $z_0 \in G$. Let $K = \overline{B(z_0,r)} \subset\subset G$ and let $M > 0$ s.t. $\mu(f)(z) \leq M$ for $z \in K$. Let $P = [z_0, z] \subseteq K$ and assume $f \in \mathcal{F}$ has no poles in P. Choose $\varepsilon > 0$ and let $z_1, \ldots, z_n = z$ be a partition of P



such that for $j=1,\dots,n$:

$$\left| \frac{1+|f(z_{j-1})|^2}{(1+|f(z_{j-1})|^2)^{1/2}(1+|f(z_j)|^2)^{1/2}} - 1 \right| < \varepsilon \quad (1)$$

and

$$\left| \frac{f(z_j)-f(z_{j-1})}{z_j - z_{j-1}} - f'(z_{j-1}) \right| < \varepsilon \quad (2)$$

This can be done by uniform cont of f, f' on P. Let $1 \le \alpha_j^2 = (1+|f(z_{j-1})|^2)(1+|f(z_j)|^2)$

$$d_\infty(f(z_0), f(z)) \le \sum_{j=1}^{n} d_\infty(f(z_{j-1}), f(z_j)) =$$

$$2\sum_{j=1}^{n} \frac{|f(z_j)-f(z_{j-1})|}{\alpha_j} \le$$

$$2\left(\sum_{j=1}^{n} \frac{1}{\alpha_j} \left| \frac{f(z_j)-f(z_{j-1})}{z_j - z_{j-1}} - f'(z_{j-1}) \right| |z_j - z_{j-1}| \right.$$

$$\left. + \sum_{j=1}^{n} \frac{|f'(z_{j-1})|}{1+|f(z_{j-1})|^2} \cdot \frac{1+|f(z_{j-1})|^2}{\alpha_j} \right) \quad (3)$$

By (2), $\displaystyle\sum_{j=1}^{n} \frac{1}{\alpha_j}\left|\frac{f(z_j)-f(z_{j-1})}{z_j-z_{j-1}} - f'(z_{j-1})\right| |z_j - z_{j-1}|$

$$\le \varepsilon \sum_{j=1}^{n} \underbrace{\frac{1}{\alpha_j}}_{\le 1} |z_j - z_{j-1}| \le \varepsilon \underbrace{\sum_{j=1}^{n} |z_j - z_{j-1}|}_{= |z_n - z_0|} = \varepsilon |z - z_0| \qquad (4)$$

And

$$2\sum_{j=1}^{n} \frac{|f'(z_{j-1})|}{1+|f(z_{j-1})|^2} \, \frac{1+|f(z_{j-1})|^2}{\alpha_j} |z_j - z_{j-1}|$$

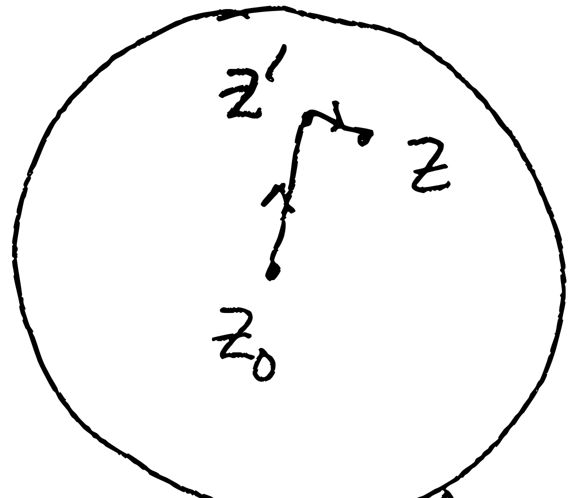
$$\sum_{j=1}^{n} \mu(f)(z_{j-1}) \left| \frac{1+|f(z_{j-1})|^2}{(1+|f(z_{j-1})|^2)^{1/2}(1+|f(z_j)|^2)^{1/2}} - 1 \right| |z_j - z_{j-1}|$$

$$+ \sum_{j=1}^{n} \mu(f)(z_{j-1}) |z_j - z_{j-1}| \le (M\varepsilon + M)|z - z_0| \qquad (5)$$

So (3) $\Rightarrow$ $\varepsilon > 0$ arbitrary

$d_\infty(f(z), f(z_0)) \le M|z - z_0|$.

Now, if $f \in \mathcal{F}$ has a pole on $[z_0, z]$, then we proceed as follows: WLOG: z_0 is not a pole (if both z_0, z are poles, then $d_\infty(f(z), f(z_0)) = d_\infty(\infty, \infty) = 0$; if z_0 is a pole but z not, simply reverse roles.)



Pick z' close to z s.t. f has no poles on $[z_0, z']$. By previous $d_\infty(f(z_0), f(z')) \le M|z' - z_0|$. Moreover, $z' \to z$, $f(z') \to f(z) = \infty$ so $d_\infty(f(z'), f(z)) \to 0$. Also, $|z' - z_0| \to |z - z_0| \Rightarrow d_\infty(f(z), f(z_0)) \le$

$$d_\infty(f(z), f(z')) + d_\infty(f(z'), f(z_0)) \quad \Rightarrow$$

$d_\infty(f(z), f(z')) \to 0$ as $z' \to z$

$d_\infty(f(z'), f(z_0)) \le M|z' - z_0| \to M|z - z_0|$

$d_\infty(f(z), f(z_0)) \leq M \cdot |z - z_0|$ also in this case. Now, equicont. of $\mathcal{F}$ at z_0 follows immediately, which completes "$\Leftarrow$".

The converse "$\Rightarrow$" is HW. $\boxtimes$